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EDITED BY

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MOTION ON THE SPRINGS OF A CARRIAGE BODY.

BY G. GREENHILL.

IN continuation of the *Mathematical Gazette*, p. 389, September 1914, on *Linear Dynamics*, the investigation is resumed here of the rolling and pitching of the body of a carriage on the springs, such as experienced in a railway carriage, tram, or motor car, and the effect can be studied as realised; but most instructively in the motibus, where the action is felt on a more violent scale, vibrating with dynamical interest and poetry.

1. Begin by supposing the carriage body lowered down and deposited on four springs at the corners of a rectangle, sinking down vertically on the springs an equal distance c suppose, the permanent average set of the springs.

A vertical oscillation of the body will then synchronize, as on p. 393, with a simple equivalent pendulum of length c, assuming the *Law of the Spring* (Hooke's, the Linear Law).

For if the body receives an additional vertical displacement x, the upward reaction of the springs against its weight W changes on this Law to $W\left(1+\frac{x}{c}\right)$, so that the unbalanced force is $W\frac{x}{c}$ back to the position of equilibrium, as in the horizontal motion of the bob of a simple pendulum of length c, where the force for a bob of weight W is $W \sin\theta$ at an angle θ, and $\sin\theta=\frac{x}{c}$, treating the displacement x as the horizontal displacement of the bob, when the motion is small.

2. Taking the length of the pendulum which beats the second from the Act of Parliament as 39·1343 inches, or 99·4 cm, and replacing it by 40 inches or 1 metre in round numbers, then with $c=10$ inches or 25 cm for the half second pendulum, the body makes 60 vertical vibrations a minute.

At this stage it is important to distinguish between vibration (period), cycle, or revolution, against oscillation (half period), beat, or stroke; as one revolution of a crank gives two strokes to the piston, and one revolution of a conical pendulum corresponds to two beats of a plane oscillation. And a locomotive engine with two cylinders will give four puffs of steam from the chimney for each revolution of the driving wheel; that is sixteen puffs a second at 60 miles an hour, with wheels 7 feet high.

Thus the half second pendulum, 10 inches or 25 cm long, will make an oscillation, beat, or stroke in half a second, and 120 in one minute; but it

makes one vibration, cycle, or revolution, as in a conical motion, in the period of one second, with a frequency of 60 to the minute.

But a 40 inch pendulum, which beats the second, revolves as a conical pendulum in two seconds.

It is prudent to be always on the guard against this confusion; thus one of our tuning forks with a frequency marked 256 per second would be marked 512 in France; and on an electrical frequency meter made in Germany 100 Polwechsel is marked, where 50 would be registered in England.

There is the analogous danger of confusion between radius and diameter, resulting on one well-known occasion in the loss of the Admiral's battle-ship.

For any other pendulum length c, in inches, the body makes

$$N=60\sqrt{\frac{10}{c}} \text{ cycle-vibrations per minute,}$$

or $c=10\left(\frac{60}{N}\right)^2$ is the length to make N vibrations.

Thus with $c=5$, $N=84$ to 85; $c=2{\cdot}5$, $N=120$.

The motion of a motor car is found unpleasant if N exceeds 100, and the springs are said to be too hard; 75 is given as a good average number for N, making $c=10\left(\frac{60}{75}\right)^2=6{\cdot}4$ inches, or 16 cm, as the average set of the springs.

3. Assimilating the vertical motion of the carriage body to a piston actuated by a crank in uniform motion, where the obliquity of the connecting rod is ignored, or else the connecting rod is suppressed, as in a certain form of feed pump, the crank describes one revolution while the piston makes two strokes, and the maximum piston velocity is at the middle of the stroke, and equal to the crank velocity; thence

$$(1) \qquad \frac{\text{average piston velocity}}{\text{maximum piston velocity}}=\frac{\text{two diameters}}{\text{one circumference}}=\frac{2}{\pi},$$

$$(2) \qquad \text{average piston velocity}=\frac{\text{maximum velocity}}{\frac{1}{2}\pi},$$

a simple rule for calculating the time of movement to rest against a resistance increasing as the displacement, analogous to the rule of the arithmetic mean for average velocity when the resistance is uniform; this rule was employed in the *Mathematical Gazette*, p. 415, for calculating the time of contact of spring buffers during the impact.

There is also the vertical vibration to be considered of a passenger on the spring cushion, and its influence on the vertical vibration of the body, in or out of tune. And when a passenger steps out, of weight P, the total recovery upward of the four springs is $4\frac{P}{W}c$.

4. When the carriage body rolls through a small angle θ, it is supposed to turn about a longitudinal axis through O (Fig. 1) midway between S and S' in the line through the top of the springs on each side; and here the moment of inertia (M.I.) intervenes of the body.

As explained in Maxwell's *Matter and Motion*, page 102, a rigid body in uniplanar motion is equivalent kinetically to a pair of particles.

The position of one particle is arbitrary, so place it at O, with the other one at P, called the *centre of oscillation* with respect to O the *centre of suspension*.

The two particles are connected by a thread or light rod OP, of fixed length l, and their weights are so adjusted that the sum is the weight W of the body, and their centre of gravity (C.G.) is at G, the C.G. of the body; and further, their distance l is taken so that the M.I. of the two particles about G is Wk^2, the M.I. of the body.

The two particles then form a system which behaves in the same way as the body, under the same applied force; and to secure these conditions, denoting OG by h, the particle at P and O will have weight

$$W\frac{h}{l}, \text{ and } W\left(1-\frac{h}{l}\right),$$

and M.I. about G

(1) $$W\frac{h}{l}(l-h)^2+W\left(1-\frac{h}{l}\right)h^2=W(lh-h^2)=Wk^2, \text{ when } l=h+\frac{k^2}{h},$$

giving the well known expression for l the length of the simple equivalent pendulum (S.E.P.).

For if the carriage body was turned upside down with G vertically below O, the body would swing about the axis through O like the simple pendulum OP, composed of a plumb bob at P and a fine thread OP.

Thus k^2 could be determined experimentally by hanging the body, or a model of it, upside down, and measuring the length l of OP which synchronizes in oscillation; and then, if h is measured too,

(2) $$k^2=lh-h^2.$$

5. Thence generally, if a rigid body, with M.I. WK^2, lb-ft², about a fixed axis in it, is urged back again by a couple $WH\sin\theta$, ft-lb, when displaced through an angle θ from a position of equilibrium, it will oscillate like a simple pendulum of length

(1) $$\frac{WK^2}{WH}=\frac{K^2}{H}, \text{ feet.}$$

Take the case of a door on hinges, not quite plumb but at an angle β from the vertical; the restoring couple is $Wh\sin\beta\sin\theta$, and the length of the equivalent pendulum is

(2) $$L=\frac{K^2}{h\sin\beta}=\left(h+\frac{k^2}{h}\right)\operatorname{cosec}\beta=l\operatorname{cosec}\beta,$$

so that the S.E.P. length l is stretched by the factor cosec β, and can be made as large as required by a diminution of β; this is the principle of the horizontal pendulum so called, required for the measurement of a very small disturbance of gravity, due to the Moon for instance.

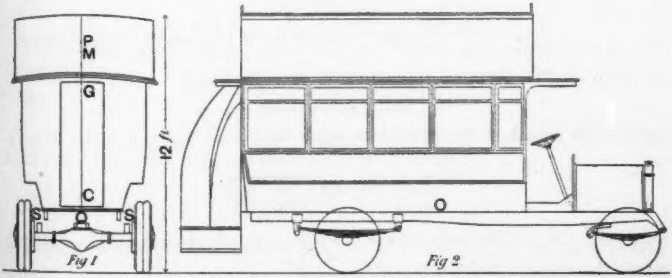

Fig 1 Fig 2

6. In Fig. 1 of the end view of a motibus, taken from Mr. Worby Beaumont's paper on the "Petrol Motor Omnibus," *Proc. I.M.E.*, March 1907, the body rolls about a longitudinal axis through O; and at an angle θ from

the upright position of equilibrium, the upward force of the springs at S and S', a distance b apart, changes to

$$\tfrac{1}{2}W\left(1 \pm \frac{\frac{1}{2}b\tan\theta}{c}\right),$$

supposing each spring always to act in the same line at right angles to the axle.

The restoring couple of the springs is then

$$(1) \qquad \tfrac{1}{2}W\left(1+\frac{\frac{1}{2}b\tan\theta}{c}\right)\tfrac{1}{2}b - \tfrac{1}{2}W\left(1-\frac{\frac{1}{2}b\tan\theta}{c}\right)\tfrac{1}{2}b = W\frac{\frac{1}{4}b^2}{c}\tan\theta, \text{ ft-lb};$$

and the upsetting couple of the weight W through the C.G., at a height h above O, is $Wh\sin\theta$; so that the total restoring couple is

$$(2) \qquad W\left(\frac{\frac{1}{4}b^2}{ch\cos\theta}-1\right)h\sin\theta, \text{ ft-lb};$$

and when θ is small, and $\cos\theta$ replaced by unity, the body rolls like a pendulum of length

$$(3) \qquad L=\frac{W(h^2+k^2)}{W\left(\frac{\frac{1}{4}b^2}{c}-h\right)}=\frac{l}{\frac{\frac{1}{4}b^2}{ch}-1}, \text{ or } \frac{l}{L}=\frac{\frac{1}{4}b^2}{ch}-1=\frac{GM}{OG}, \; L=\frac{OG \,.\, OP}{GM},$$

on marking off $OM=\frac{\frac{1}{4}b^2}{c}=\frac{OS^2}{OC}$ on OP, by making the angle $OSM=OCS$.

Then G must be below M for L to be positive, and the upright position stable; and the motion becomes slow and easier as L is increased by raising G closer to M; and the body swings like a free pendulum as if suspended from an axis through O', where

$$(4) \qquad GO'=\frac{GP \,.\, GM}{OP}.$$

7. But with G above M, and h a little greater than $\frac{\frac{1}{4}b^2}{c}$, the body will loll over to one side, at an angle θ, where

$$(1) \qquad \cos\theta=\frac{\frac{1}{4}b^2}{ch}.$$

At a small extra inclination ϕ, the total restoring couple is

$$(2) \qquad W\frac{\frac{1}{4}b^2}{c}\tan(\theta+\phi)-Wh\sin(\theta+\phi)$$
$$= W\left(\frac{\frac{1}{4}b^2}{c}\sec\theta-h\right)\sin\theta + W\left(\frac{\frac{1}{4}b^2}{c}\sec^2\theta-h\cos\theta\right)\sin\phi,$$

of which the first term vanishes, and the second term is

$$(3) \qquad W(\sin\theta-\cos\theta)h\sin\phi;$$

making the length of the equivalent pendulum

$$(4) \qquad L=\frac{l}{\sec\theta-\cos\theta}=\frac{l}{\frac{ch}{\frac{1}{4}b^2}-\frac{\frac{1}{4}b^2}{ch}},$$

where $\frac{\frac{1}{4}b^2}{c}<h$, and the body lolls to one side; against the value of L is (3), § 6, where the upright position is stable, and $\frac{\frac{1}{4}b^2}{c}>h$.

The Leaning Tower of Pisa may be supposed to have lost the stability of the upright vertical position in a similar way, the foundation flinching as if supported on springs, or as if buoyed up by fluid earth.

8. Suppose the body upright, and a passenger P to move across a distance x; the C.G. will move the reduced distance $\frac{W}{P}x$ in the same direction, and a heeling couple Px is applied, which tilts the body to an angle θ, where it is balanced by the restoring couple

$$W\left(\frac{\frac{1}{4}b^2}{c}-h\right)\sin\theta;$$

so that the body is heeled at one in

(1) $$\frac{1}{\sin\theta}=\frac{W}{P}\frac{\frac{\frac{1}{4}b^2}{c}-h}{x};$$

but the time of oscillation will not be affected to an appreciable extent, as due to a term involving $\sin\theta$, which is insensible.

The plumb bob can be shown inside in Fig. 2, on the thread CO of length c, to beat time with the vertical oscillation; and it may be replaced by a sheet of cardboard, a square doubled along a diagonal into half a square, like a set square, an isosceles right-angled triangle ABC of height $OC=c$, suspended at the centre O.

9. Rounding a curve of radius R feet, with velocity V f/s, the horizontal centrifugal force (C.F.) $W\frac{V^2}{gR}$, lb, acting through the C.G., is shared equally by a horizontal reaction of the four springs, and will heel the body through an angle θ, such that the moment round O of the C.F., $W\frac{V^2h}{gR}$, ft-lb, is counterbalanced by the restoring couple, or

(1) $$W\frac{V^2h}{gR}=W\left(\frac{\frac{1}{4}b^2}{c}-h\cos\theta\right)\tan\theta,$$

giving the body a slope of one in

(2) $$\frac{1}{\tan\theta}=\left(\frac{\frac{1}{4}b^2}{ch}-\cos\theta\right)\frac{gR}{V^2},$$

in which $\cos\theta$ may be replaced by unity; the effect being practically the same as if a passenger P had moved across a distance x feet, such that

(3) $$Px=W\frac{V^2h}{gR}.$$

The weight taken off the outer pair of wheels and transferred to the inner pair is

(4) $$P\frac{x}{b'}=W\frac{V^2h'}{gRb'},$$

with a wheel track b' feet wide, and h' the height of the C.G. above the road; and the outer wheels would begin to lift when

(5) $$W\frac{V^2h'}{gRb'}=\tfrac{1}{2}W,\quad \frac{V^2}{2g}=\tfrac{1}{4}R\frac{b'}{h'}.$$

But the skidding coefficient is $\mu=\frac{\text{C.F.}}{W}=\frac{V^2}{gR}$, and if $\mu<\frac{b'}{2h'}$, the car will skid before toppling, in turning a sharp corner too fast.

10. The slope in (2), § 9, is relative to the road; but to a passenger, with an effective level at a slope of one in $\frac{gR}{V^2}$ of the plumb line, the apparent slope

θ of the body of the carriage will be increased by this slope ϕ of the plumb line or spirit level, giving a total slope of $\phi+\theta$ with respect to the seat of the carriage; and as ϕ and θ are small, we can put

(1) $$\tan(\phi+\theta)=\tan\phi+\tan\theta=\frac{V^2}{gR}+\frac{1}{\frac{\frac{1}{4}b^2}{ch}-1}\frac{V^2}{gR}=\frac{1}{1-\frac{ch}{\frac{1}{4}b^2}}\frac{V^2}{gR},$$

a slope of one in

(2) $$\left(1-\frac{ch}{\frac{1}{4}b^2}\right)\frac{gR}{V^2},$$

as it seems to a straphanger.

11. Taking the body as quite plumb when the road is level, then standing on a camber or banking of a small angle a, a slope of one in n, will give an additional tilt of the body on the springs through θ, where the tilting couple $Wh\sin(a+\theta)$ is to be equated to the righting couple of the springs $W\frac{\frac{1}{4}b^2}{c}\tan\theta$, so that

(1) $$\tan\theta=\frac{\tan a}{\frac{\frac{1}{4}b^2}{ch}-1},\quad \tan a+\tan\theta=\frac{\tan a}{1-\frac{ch}{\frac{1}{4}b^2}},$$

a slope of the body of

$$\text{one in }\left(\frac{\frac{1}{4}b^2}{ch}-1\right)\text{ with respect to the road, one in }\left(1-\frac{ch}{\frac{1}{4}b^2}\right)$$

with respect to the spirit level; and, as before in § 9, the time of a rolling oscillation will be unaffected to any sensible extent.

12. The treatment is the same for the pitching oscillation of the body about a transverse axis through O below G, in the plane through the top of the springs (Fig. 2).

But in the motibus the C.G. vertical does not fall midway between the axles, and the load is not distributed equally between the wheel pairs, as seen indicated on the off side of the body, painted FAW and HAW for front and hind axle weight, with UW for unloaded weight, and speed in miles an hour on the other (the near) side, as 12 MPH; for instance, FAW 2 tons, HAW 4 tons, UW 3 tons 9 cwt., the tare weight, leaving 2 tons 11 cwt. for net load of 34 passengers, at an average of 12 stone each. And c is taken to change in the ratio of the tare UW to the gross weight.

Denoting the weight carried by the front and hind pair of springs by W_1 and W_2, $W_1+W_2=W$; and by a_1, a_2, the distance from the axis through O, $a_1+a_2=a$, the wheel base,

(1) $$W_1a_1=W_2a_2=W\frac{a_1a_2}{a}.$$

The strength of the springs is supposed adjusted so that they are all depressed through the same vertical distance c when the body is deposited on them; as a vertical oscillation can then take place without calling up a pitching motion as well.

If the trim of the body by the head changes through a small angle θ about the transverse axis through O, the restoring couple of the springs to the upright is

(2) $$W_1a_1\left(1+\frac{a_1\tan\theta}{c}\right)-W_2a_2\left(1-\frac{a_2\tan\theta}{c}\right)$$
$$=\frac{W_1a_1^2+W_2a_2^2}{c}\tan\theta=W\frac{a_1a_2}{c}(\tan,\text{ or }\sin)\,\theta,\text{ ft-lb},$$

to be diminished by the tilting couple $Wh \sin \theta$, ft-lb, to obtain the net restoring couple ; so that the body pitches like a pendulum of length

(3) $$L_1 = \frac{W(h^2+k_1^2)}{W\frac{a_1a_2}{c} - Wh} = \frac{l_1}{\frac{a_1a_2}{ch} - 1},$$

or

(4) $$\frac{l_1}{L_1} = \frac{a_1a_2}{ch} - 1, \quad l_1 = \frac{h^2+k_1^2}{h},$$

l_1 the equivalent pendulum of the body, swinging upside down about the transverse axis through O.

With $a_1 = a_2 = \frac{1}{2}a$, this reduces to

(5) $$\frac{l_1}{L_1} = \frac{\frac{1}{4}a^2}{ch} - 1,$$

as before in (3), § 6, for rolling.

13. In a sudden start from rest when the clutch is inserted, and the body W, together with M, spring-axle-wheel weight in lb, is urged forward with a force X pounds at the bite of the driving wheels on the ground, the body starts to move like the equivalent particles at O and P, $W\left(1-\frac{h}{l}\right)$ and $W\frac{h}{l}$; so that before the springs have time to act, P is left behind, and it is only the weight $W\left(1-\frac{h}{l}\right)$ at O, together with M, which is urged by the force X, and move bodily together with acceleration

(1) $$\frac{X}{W\left(1-\frac{h}{l}\right)+M}g;$$

and this exceeds the acceleration $\frac{X}{W+M}g$ acquired by a rigid body without springs by

(2) $$\left[\frac{1}{W\left(1-\frac{h}{l}\right)+M} - \frac{1}{W+M}\right]Xg = \frac{XW\frac{h}{l}}{\left[W\left(1-\frac{h}{l}\right)+M\right](W+M)}g,$$

equivalent to a diminution $W\frac{h}{l}$ of effective inertia or weight, so that the flexibility on the springs improves the ease and rapidity of starting ; and the flexibility of the passengers too contributes to the effect, and illustrates the difference of live and dead load. Additional flexibility to forward motion can also be given by horizontal springs to the driving axle, allowing it to move under the frame. And the spring cushions help to attenuate the vertical vibration.

14. The effect was well known in the old sailing ship, cramped and deadened, when stowed tight with cargo, but working itself free in the sea way gradually, as described by the old Sailor in *Two Years before the Mast.* "Stand by! you'll see her work herself loose in a week or two, and then she'll walk up to Cape Horn like a racehorse."

It is a general principle in Dynamics that, in giving motion to a body, the greater flexibility absorbs least energy ; as with a cat in jumping.

And a flexible chain, whirled round an axis with given angular momentum, will assume the form of least energy, and so of greatest moment of inertia ;

and the form is determined thence by the Calculus of Variations. Working it out for the chain in one plane, like a skipping rope, the curve is given by the elliptic function sn-oidal equation.

15. In a motor lorry, not required to start and stop so frequently, the pull X is applied generally through the medium of a chain, as on a bicycle, instead of a Cardan joint and gearing; the chain works over a toothed wheel on the driving axle; and if geared down so that the chain moves past the frame, like a bicycle chain past the body, with $\frac{1}{n}$ of the velocity of the wheels over the ground, the tension difference over the toothed wheel or net driving tension is nX.

16. As X continues, the body sways on the springs and would continue to oscillate unless the spring action was damped.

When the damping is complete, the total weight $W+M$ lb will move bodily together with acceleration

$$(1) \qquad f=\frac{X}{W+M}g\,;$$

and the front of the body rises and the rear sinks on the springs over the axles, until balanced by the restoring moment of the springs; and so the floor will slope at one in

$$(2) \qquad \left(\frac{a_1a_2}{ch}-1\right)\frac{g}{f}=\left(\frac{a_1a_2}{ch}-1\right)\frac{W+M}{X}$$

with the ground; but to a passenger straphanger, spirit level, or short plumb line, the slope will appear to be one in

$$(3) \qquad \left(1-\frac{ch}{a_1a_2}\right)\frac{g}{f}=\left(1-\frac{ch}{a_1a_2}\right)\frac{W+M}{X}$$

with respect to his apparent floor level, disturbed to a slope of one in $\frac{g}{f}$ with the ground.

17. At full speed, when the acceleration is zero, the body resumes gradually the ordinary level on the springs; and the starting effects are reversed when the break is put on to stop, producing retardation r.

And when the motibus stops, gravity resumes its normal vertical, and a jerk is felt by the change of direction, in addition to the movement of the floor to a parallel with the ground.

The jerk felt on the Tube railway, coming after the stop, will tend to throw a passenger on his back or face according as he walks out by the front or rear of the car.

18. Damping then in the springs is required to a moderate extent, to prevent the motion of the body from continuing lively too long.

The damping in flat-plate springs arises in the friction of the plates slipping over each other; and when the friction becomes excessive the plates should be oiled ("Springs," by G. H. Baillie, in the *Proceedings of the Institution of Automobile Engineers*, May 1913).

In a locomotive engine the springs over the driving wheels require to be more lively than over carrying wheels, so as to keep up a better bite on the rail; but if the other carrying springs were allowed to be too lively, the motion of oscillation might increase to the dangerous extent, and the engine leave the rails.

So too, without the damping, the body of the motibus, starting from rest with acceleration f, would continue to oscillate through the angle $2\tan^{-1}f/g$.

19. It is usual in a mathematical investigation of damping to assume that the friction in a small oscillation is represented by a term proportional to the relative velocity of slipping, as this assumption makes the analysis tractable, and it gives a good representation of the facts, provided the relative velocity is low.

The damped vibration is then represented by a periodic term of which the amplitude is qualified by a factor decreasing at compound discount, and Naperian logarithmic decrement is the name given to one-hundredth of the rate per cent per period of vibration at which the amplitude of vibration diminishes at continuous compound discount.

In a graphical representation the effect is to change the circle in Fig. 2, p. 392, *Math. Gazette*, into an equiangular spiral, as explained in Maxwell's *Electricity and Magnetism*, § 731.

20. Consider a symmetrical motor car, where the four wheels are equal for interchangeability, and $a_1 = a_2 = \frac{1}{2}a$, going slowly over the crest of a road of two equal inclines sloping at angle α, and meeting in a ridge, say over an old canal bridge.

As the front wheels pass over the ridge, the C.G. of the car will proceed to describe an ellipse, with horizontal and vertical axis $2h + a \cot \alpha$, $2h - a \tan \alpha$.

Thus, if $h = \frac{1}{2}a \tan \alpha$, the C.G. will advance in a horizontal line for a distance $a(\tan \alpha + \cot \alpha) = 2a \operatorname{cosec} 2\alpha$; and if $h < \frac{1}{2}a \tan \alpha$, the car can remain on the crest in stable equilibrium, and requires to be drawn off.

Conversely at a dip in the road.

Going fast up the incline at full speed S m.p.h., the front wheels would leave the ground at the ridge; and treating the car as a particle, it would proceed to describe a parabola, and lose contact with the road for a length

$$0{\cdot}09 S^2 \sin \alpha \text{ yards, and for } 0{\cdot}18 S \sin \alpha \text{ seconds,}$$

and come down again at one in $\frac{1}{2}(\cot \alpha + 3 \tan \alpha)$ with the ground.

Up and down one in 20, for instance, at 30 m.p.h., the car will leap 1·8 yards in 0·18 second, and come down on the ground at one in 10.

G. Greenhill.

THE DISSECTION OF RECTILINEAL FIGURES.

By W. H. Macaulay, M.A.

(*Concluded.*)

Another function of the pentagon dissection, Fig. 19, is to provide, by means of an additional cut in each pentagon, an eight-part type of dissection of a pair of quadrilaterals. This, which I will call the first type of eight-part dissection of quadrilaterals, is shown in Fig. 24. The dissection of the pair of quadrilaterals is shown in full lines, and dotted lines are added to show a pair of auxiliary pentagons, the lettering being the same as in the previous figures. A dissection of the pair of pentagons is shown, and there is an additional cut in the first pentagon joining T to the middle point X of the base AB, and an additional cut in the second pentagon joining W to the middle point Y of the base CD. The other new lines shown in the figure are merely consequential on the formation of a pair of quadrilaterals by means of these two cuts. There is no relation between the dimensions of the quadrilaterals except the equality of their areas. Each quadrilateral is divided by six lines, two or more of which are broken; four of these lines are half sides of the other quadrilateral, and the other two are half diagonals, one of one quadrilateral and the other of the other quadrilateral. When there are eight parts, as shown

in Fig. 24, two lines in each quadrilateral are broken, so that each is made up of two parallel portions. A dissection of this type with, when necessary, more parts and more broken lines, is applicable to any pair of quadrilaterals of equal area (excluding quadrilaterals whose sides cross). The dissection depends, as before, on the existence of the identical parallelograms $HLNG$ and $JKMO$; each has one side equal to a half diagonal of one quadrilateral, and another side equal to a half diagonal of the other quadrilateral, and area equal to half that of a quadrilateral. So the question of the universal applicability of the dissection depends on whether it is always possible to construct such a parallelogram, which I will call, in order to give it a name, a "joint parallelogram." Any one joint parallelogram implies the existence of another

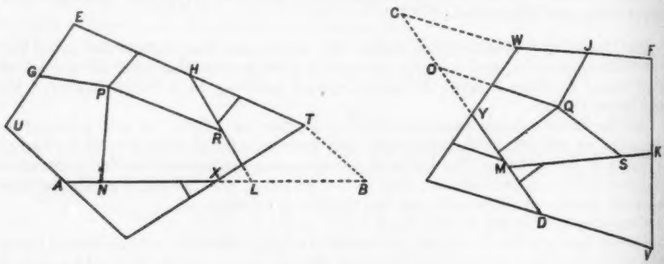

FIG. 24.

one, namely, one with the same sides, but with acute and obtuse angles interchanged. There are exceptional cases in which these parallelograms coincide, each being a rectangle; so for the purpose of counting, let us either disregard rectangles, or count a rectangle as two coincident parallelograms. Let us also, for the purpose of counting, suppose the lengths of the diagonals to be all different. We can certainly find a joint parallelogram for any given pair of quadrilaterals, for all that we have to do is to combine the longest of the four diagonals with either of the diagonals of the other quadrilateral. In this way we get four joint parallelograms in every case; and we get four more, that is to say, eight altogether, if one of the quadrilaterals has both its diagonals longer than either diagonal of the other; this is the greatest possible number. If the quadrilateral which has the longest diagonal has also the third in order of magnitude, we get either six or eight joint parallelograms. If one quadrilateral has the longest and the shortest diagonal, we get either four or six or eight. In any case the number can be found at once if the lengths of the diagonals and the area of the quadrilaterals are given. It should be noted that four different pairs of pentagons can be formed by cutting the quadrilaterals along the lines AX and DY, but that these all give the same dissections of the quadrilaterals; accordingly, the number of distinct dissections derivable from a joint parallelogram is eight. Thus a pair of quadrilaterals possesses either 32 or 48 or 64 distinct dissections of the type in question, according to whether the number of joint parallelograms is four or six or eight. It is possible for a large proportion of these dissections to have eight parts, but I do not know how many. In the case of the pair of quadrilaterals shown in Fig. 24, six of the dissections derived from the joint parallelogram $HLNG$ have eight parts. This pair of quadrilaterals has four joint parallelograms.

This type of dissection of a pair of quadrilaterals may be analysed in another way. Any quadrilateral has a parallelogram, of half its area, formed by joining the middle points of its sides; let us call this its "core." It has also two points,

which I will call its "index points," which are such that radii from an index point to the angles of the core are parallel to the sides of the quadrilateral. The index points are without or within the core, according as the quadrilateral has or has not a re-entrant angle; the centre of the core bisects the line joining them. A core and one index point, given with reference to the core, specify a quadrilateral, subject to one further choice as to which of two positions with reference to the core it occupies. To dissect a given pair of quadrilaterals, draw their cores and index points, and select a joint parallelogram, say $ABCD$, Fig. 25. Then there are two attitudes which the cores can

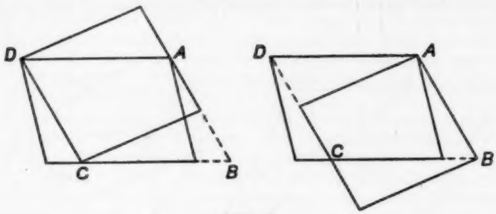

FIG. 25.

assume, such that each core has a side in common with $ABCD$ and is between the same parallels. These two attitudes will serve to determine all the dissections derivable from this joint parallelogram. They are shown in Fig. 25, the cores being drawn in full lines, and the joint parallelogram completed with dotted lines. The method may be seen by reference to Fig. 24. Here, as it happens, the first attitude is used for the first quadrilateral; the joint parallelogram is $HLNG$, the core of the first quadrilateral is $HXAG$, and the attitude determines a position for the second core with three angular points at R, N, G, then P is one of the index points of the second core; thus a dissection is determined, subject to necessary provision for broken lines. The corresponding dissection of the second quadrilateral is similarly connected with the second attitude. To dissect the first quadrilateral, we may employ either of the two attitudes, and having chosen one of them we may place the first core in either of two positions with its corners at the middle points of the sides of the first quadrilateral, and we may then take for the point P either of the two index points of the second core; these three choices combined give eight alternative dissections of the first quadrilateral, and these are the eight dissections which are derived from the joint parallelogram in question. The point P may happen to fall outside the quadrilateral, complicating the arrangement of broken lines. The dissections involve a considerable variety of geometrical constructions, though they are all governed by one scheme.

Another type, which I will call the second type, of eight-part dissection of a pair of quadrilaterals, is shown in Fig. 26. It depends on the construction of a parallelogram, of area equal to that of a quadrilateral, with a side equal to a diagonal of one quadrilateral and another side equal to a diagonal of the other; this I will call a joint parallelogram of the second type. In Fig. 26 a pair of quadrilaterals and one of their parallelograms are placed so that the sides of the parallelogram produced would bisect sides of the quadrilaterals. A four-part dissection of one of the quadrilaterals and the parallelogram is made by drawing AB, CD bisecting sides of the quadrilateral, and placing between AB and CD a line EF equal to the selected diagonal of the other quadrilateral. Similarly, a four-part dissection of the other quadrilateral and the parallelogram is made by means of lines GH, JK, LM. So far as these two dissections are concerned, each of the lines EF and LM has liberty to take any one of a series of parallel positions; a definite position will now be

chosen for each of them as follows. The first dissection of the parallelogram is made by placing the core $ACDB$ of the first quadrilateral in some such position as $NOPQ$, between parallel sides of the parallelogram, and marking on it an index point R. Similarly, the second dissection is made by placing the core of the other quadrilateral in some such position as $STUV$, between the other pair of sides of the parallelogram, and marking on it an index point. We can now slide these cores, each with an index point attached to it, between the parallel lines till the two points coincide; the two dissections then combine so as to give the eight-part dissection of the pair of quadrilaterals. In the figure the common index point, R, is an intersection of diagonals of the quadrilaterals. It is clear that the two auxiliary dissections of the quadrilaterals

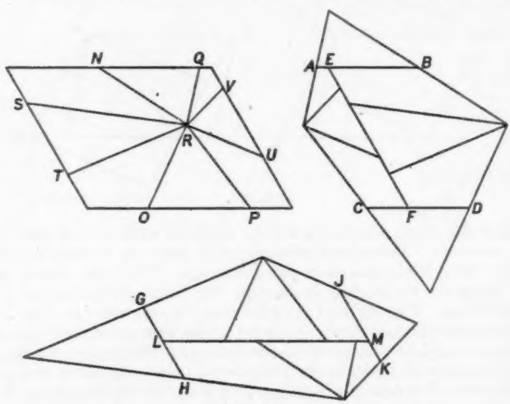

FIG. 26.

and the parallelogram always exist, though they will involve more parts and broken lines if any of the lines AB, CD, GH, JK have to be produced in order that EF and LM may be placed between them; therefore the resulting dissection of the quadrilaterals must exist, though it may have broken lines and consequently more than eight parts. I propose, however, not to count as belonging to this type of dissection those dissections which are derived from a parallelogram with the common index point R outside it. Such a case cannot occur if the quadrilaterals have no re-entrant angles; so let us deal first with this case, in which the index points are within their respective cores. The condition for a dissection with only eight parts is then that the cores, with index points coinciding, can be placed between the sides of the parallelogram, so as to be wholly within it. Any pair of quadrilaterals, of equal area, must possess either four or six or eight joint parallelograms of the second type. Choose one of these parallelograms; then the cores, with index points coinciding, can assume two different attitudes between the parallel lines. Each of these attitudes gives a dissection, so there are either eight or twelve or sixteen dissections of the type in question, though not many of them can actually have only eight parts. The pair of quadrilaterals in Fig. 26 has sixteen dissections of the second type, of which only one has only eight parts; it has also one dissection of the first type with eight parts. A simple way of drawing pairs of quadrilaterals with at least one dissection of the second type with eight parts is to start with a parallelogram. The choice of a parallelogram involves three quantities, the choice of positions of the segments of its sides involves four more, and the choice of a position for R

involves two more; nine quantities altogether. This is the number of independent variables involved in the construction of a pair of quadrilaterals of equal area.

If the quadrilaterals have re-entrant angles, causing index points to lie outside their cores, the common index point R may still be within the joint parallelogram, and in those cases we have the same dissections as before, and the same condition for a dissection with exactly eight parts. But the existence of cases in which the common index point falls outside the parallelogram makes the method which has been used for counting dissections no longer applicable. And it seems clear that pairs of quadrilaterals exist which have no dissection of the type in question, as I have defined it.

It should be noted that a pair of pentagons, each with two sides equal and parallel, has, if the bases are equal, a three-part dissection, corresponding to the three-part dissection of a pair of triangles with a common side (see Fig. 1, p. 381). This can be seen by drawing a pentagon of this type, choosing any point within it, and joining this point to any point of the base and to the middle points of the sides opposite the base. We thus get three parts which can be rearranged to form another pentagon with the same base and area. The choices of points introduce three independent variables, which is the number involved in the construction of the pentagons. From this dissection various other dissections may be obtained. By one cut one of the pentagons is converted into a quadrilateral, which thus has a four-part dissection with the other pentagon, whose base is equal to a diagonal of the quadrilateral. The particular case in which this second pentagon is a triangle gives Mr. Taylor's four-part dissection of a quadrilateral and a triangle with a side equal to a diagonal of the quadrilateral, which has already been mentioned. The particular case in which the second pentagon is a quadrilateral with two parallel but unequal sides gives some further results by means of a single cut through the middle point of its base. We may take this cut so as to make a triangle, and thus get a six-part dissection of a quadrilateral and a triangle with no coincidence of dimensions. Or we may take it so as to make a quadrilateral with two sides parallel, but with no coincidence of dimensions with the first quadrilateral, and thus get a six-part dissection of these two figures. As a particular case of this, we get one of the several types of six-part dissection of a quadrilateral and a parallelogram or square.

The first type of eight-part dissection of a pair of quadrilaterals may be regarded as a particular case of an exactly similar eight-part dissection of a pair of pentagons, each with two sides parallel but not equal. This dissection is obtained by substituting for the cut XT in Fig. 24 a cut along a line through X intermediate between XT and XB, and for the cut YW one intermediate between YW and YC. This dissection of a pair of pentagons is the general case of the eight-part dissection in question. It leads by a fairly obvious procedure to a twelve-part dissection (with the usual extensions) of what may be called a general pair of pentagons, if we only mean by this a pair of pentagons with their sides unequal and none of them parallel. But it does not apply to all pairs of pentagons, for it is based on the construction of a joint parallelogram which does not always exist. Each pentagon has five diagonals, and it is necessary to construct a parallelogram of area equal to half that of a pentagon, and with one pair of sides equal to a half diagonal of one pentagon, and the other pair equal to a half diagonal of the other. Thus the area of a pentagon must not be greater than half that of the rectangle contained by the longest diagonal of each. For example, this dissection is not applicable to a pair of pentagons each of which approaches to being regular.

It will be seen that all the dissections obtained so far for pairs of figures with no coincidence of dimensions, which appear to secure the greatest possible economy of parts, are either particular cases of, or are immediately derivable from three types of dissection, which I will call radical types. These are

the three-part dissection of a pair of parallelograms shown in Figs. 10 and 11, p. 386; the four-part dissection of a pair of pentagons, each with two sides equal and parallel, shown in Fig. 19; and the second type of eight-part dissection of a pair of quadrilaterals shown in Fig. 26. The various developments of these dissections afford a remarkable variety of geometrical constructions, by no means fully indicated by the figures which I have given.

Mr. H. W. Richmond kindly allows me to publish the diagram shown in Fig. 27, by means of which he has tabulated the dissections with four parts possessed by a triangle and a square, of the first, second and third types. Other types of dissection may exist, but these are all that we know at present. The diagram is accurately drawn to scale. The length of the line HK can be checked by means of the fact that it is equal to its distance from the straight line bounding the area t, and equal to half its distance from the straight line bounding C, D and E. The distance of HK from the straight line bounding g is $\frac{2}{3}HK$. All the dimensions of the diagram are multiples or simple fractions of HK, and all the curves are portions of eleven circles which touch the line HK. The diagram shews 31 areas which are denoted by 26 small letters and 5 capital letters. Two areas which are too small to

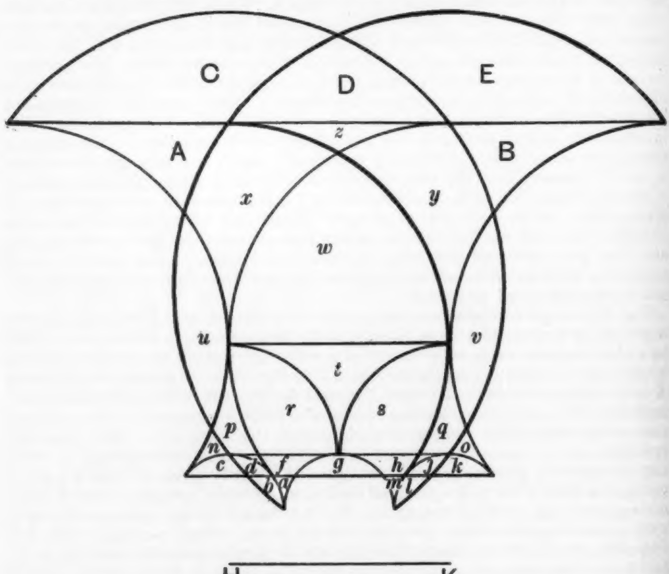

FIG. 27.

contain a letter in the figure are denoted by e and i. The shape of a triangle is specified by the position of its vertex in the diagram, HK being its base. The triangles which possess dissections with four parts are those of which the vertex lies in one or other of the 31 areas. The areas can be arranged in eight groups, such that all members of a group have the same dissections, because each contains all varieties of shapes of triangles belonging to the

group, provided that a triangle and its reflection are, for this purpose, not distinguished. Thus the triangles C have the same shapes as the triangles k, and as the triangles b; and the triangles E, c and l are the reflections of these. Three of the groups have in this way six members; but in the other groups some of the members combine so that the number is reduced to three or one.

The following table gives for each group of areas the number of distinct dissections of each type, and the total. For the purpose of this counting, each continuous series of dissections of the first type is reckoned as one dissection, not as an infinite number. The second type is Fig. 13, vol. vii. p. 387.

Group of areas.	1st type.	2nd type.	3rd type.	Total.
CkbEcl	1	0	1	2
AoaBnm	0	1	2	3
xqfyph	2	1	4	7
Ddj	2	0	2	4
zei	4	0	2	6
wrs	0	2	6	8
t	0	0	6	6
guv	2	0	2	4

Let us take any set of corresponding points in the areas of a group with six distinct members, such as $CkbEcl$, and denote the points by these letters. Then the triangles with vertices C, k, b are similar; also the triangles with vertices E, c, l are similar and are reflections of the other three. Thus HcC, KlC, Kbc, Hlk, HbE and KkE are straight lines, and the six products $Hc \,.\, HC$, $Kl \,.\, KC$ etc., are each equal to HK^2; so if we choose the point C, the other five points are determined by inversion with reference to H and K, and the boundaries of the five areas grouped with C are found from the boundaries of the area C by the same inversions.

For each type of dissection bounding lines can easily be drawn for a single dissection applicable to a limited range of triangles; and the diagram for this type can then be completed, partly by symmetry and partly by inversion, or wholly by inversion. The final diagram, Fig. 27, is then obtained by superposition of the diagrams for the three types. It is found that nearly all the lines of any one of these diagrams coincide with lines in one if not both of the other diagrams, so that the final diagram is simpler than it might be expected to be. W. H. MACAULAY.

THE DISCOVERY OF LOGARITHMS BY NAPIER.

By PROF. H. S. CARSLAW.

(*Concluded.*)

§ 11. We are now in a position to understand how Napier completed his Tables.

We take, to begin with, his *First Table.* The logarithm of the first term is zero. From § 10, Theorem I., the logarithm of the second term lies between $r-s$ and $(r-s)\frac{r}{s}$; that is, between 1·000 000 0 and 1·000 000 1. If this is taken as the A.M. of these two numbers, we have for the logarithm of the second term in that Table the value 1·000 000 05. The remaining terms in that Table form with these two a G.P., so that the logarithms have a common

difference 1·000 000 05, and the logarithm of the last term can be taken as 100·000 005.

From the logarithm of the last term in the First Table, we pass, by the aid of § 10, Theorem II., to the logarithm of the second term in the Second Table. This is found to lie between two limits, close to each other, and, approximating as before, it is given by 100·000 500 0. In this Table there are 51 terms in G.P., so that we obtain the logarithms of each of the terms easily by addition of the common difference; and, in particular, the last term has for its logarithm 5000·025 000 1.

By a process somewhat similar, it is easy to pass from the last term of the Second Table to the second term of the first column of the Third Table. The first term in that column being radius, its logarithm is zero. The logarithm of the second term is found to be 5001·250 417. This number is, therefore, the common difference of the logarithms of the terms in that column. From the last of these numbers, one passes easily to the first of the second column. Its logarithm is found to be 100 503·358. The numbers at the top of these 69 columns are in G.P., so we are now able to write down the logarithms of all of them. But the second term in each column is a number differing only slightly from the first, so its logarithm can be obtained. And when the first two logarithms for each column are known, the others follow by addition of the suitable common difference.

These are the steps taken by Napier in obtaining the logarithms of the terms in his Third Table. The results are contained in the *Radical Table*, an extract from which is now given:

The Radical Table.

First Column.		Second Column.			69th Column.	
Natural Numbers.	Logarithms.	Natural Numbers.	Logarithms.	and the others up to	Natural Numbers.	Logarithms.
10 000 000·0000	·0	9 900 000·0000	100 503·3		5 048 858·8900	6 834 225 8
9 995 000·0000	5 001·2	9 895 050·0000	105 504·6		5 046 334·4605	6 839 227·1
9 990 002·5000	10 002·5	9 890 102·4750	110 505·8		5 043 811·2932	6 844 228 3
9 985 007·4987	15 003 7	9 885 157·4237	115 507·1		5 041 289·3879	6 849 229·6
9 980 014·9950	20 005·0	9 880 214·8451	120 508·3		5 038 768·7435	6 854 230·8
etc., up to						
9 900 473·5780	100 025·0	9 801 468·8423	200 528·2		4 998 609·4034	6 934 250·8

§ 12. The numbers for which the logarithms have been found in the Radical Table form a dense enough set, lying between the radius 10^7 and approximately the half-radius, to allow with the help of § 10, Theorem II., the logarithm of the sines of angles between 30° and 90° to be calculated.

To obtain the logarithms of the sines of the angles from 0° and 30°, Napier indicates two separate methods. In the first the given sine is to be multiplied by some number 2, 4, 8, 10, 20, 40, 80, 100, 200, or any other proportional number contained in a small table he has calculated, in which the corresponding differences of the logarithms for these multipliers are given.

In the second the formula

$$\sin 2\theta = 2 \sin \theta \sin \left(\frac{\pi}{2} - \theta\right)$$

is used, but it must be noticed that this formula, in his notation, where the sines are lines in a figure of which the radius is r, must be written as

$$\frac{r}{2}\sin 2\theta=\sin\theta\sin\left(\frac{\pi}{2}-\theta\right).$$

It follows that, with Napier's logarithms,

$$\log\sin 2\theta+\log\frac{r}{2}=\log\sin\theta+\log\sin\left(\frac{\pi}{2}-\theta\right).$$

Thus the table of logarithms of sines can be extended to 15°. Then applying the same formula, it can be continued to 7° 30′, and so on indefinitely.

Napier had set out to construct a Table of the Logarithms of the Sines of Angles from 0° to 90°. We have now seen how he effected his purpose.

A page of his Table or Canon of Logarithms is given below. The middle column headed *differentiae* contains the logarithms of the corresponding tangents :

Extract from Napier's Canon of Logarithms (1614).

Gr. 30.

30 MIN.	SINUS.	LOGARITHMI.	+ DIFFERENTIAE. −	LOGARITHMI.	SINUS.	
0	5 000 000	6 931 469	5 493 059	1 438 410	8 660 254	60
1	5 002 519	6 926 432	5 486 342	1 440 090	8 658 799	59
2	5 005 038	6 921 399	5 479 628	1 441 771	8 657 344	58
⋮						⋮
29						31
30	5 075 384	6 781 827	5 292 525	1 489 302	8 616 292	30

§ 13. The discovery of "another and better kind of logarithms" had preceded the publication of the *Constructio*, and further calculations on the lines of the above Tables were unnecessary. Briggs' account of the matter is given in his *Arithmetica Logarithmica* of 1624, which contains the Table of Logarithms as we know them and use them in our own day.

"I myself," he says, "when expounding publicly in London their doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 should stand for the logarithm of the whole sine, as in the Canon Mirificus, but that the logarithm of the tenth part of the whole sine, that is to say, 5 degrees 44 minutes and 21 seconds, should be 10,000,000,000. Concerning that matter I wrote immediately to the author himself; and as soon as the season of the year and the vacation time of my public duties of instruction permitted, I took journey to Edinburgh, where, being most hospitably received by him, I lingered for a whole month. But as we held discourse concerning this change in the system of logarithms, he said that for a long time he had been sensible of the same thing and had been anxious to accomplish it, but that he had published those he had already prepared, until he could construct tables more convenient, if other weighty matters and his frail health would permit him so to do. But he conceived that the change should be effected in this matter, that 0 should become the logarithm of unity, and 10,000,000,000 that of the whole sine ; which I could not but admit was by far the most convenient of all. So, rejecting those

which I had already prepared, I commenced, under his encouraging counsel, to ponder seriously about the calculation of these tables."

Napier refers to the same point in the *Constructio* in an Appendix, entitled, "On the Construction of another and better kind of Logarithms, namely one in which the logarithm of unity is 0."

"Among the various improvements of Logarithms," he says, "the more important is that which adopts a cypher as the Logarithm of unity, and 10,000,000,000 as the Logarithm of either one-tenth of unity or ten times unity. Then, these being once fixed, the Logarithms of all other numbers necessarily follow."

He gives three methods of finding the logarithms on this scheme. We shall confine ourselves to the case where $\log 10 = 10{,}000{,}000{,}000$.

Then we have, according to his first method,

$$\left.\begin{aligned}
\log 10^{\frac{1}{5}} &= 2{,}000{,}000{,}000,\\
\log 10^{\frac{1}{5^2}} &= 400{,}000{,}000,\\
\log 10^{\frac{1}{5^3}} &= 80{,}000{,}000,\\
&\;\;\vdots\\
\log 10^{\frac{1}{5^{10}}} &= 1{,}024,\\
\log 10^{\frac{1}{2\cdot 5^{10}}} &= 512,\\
&\;\;\vdots\\
\log 10^{\frac{1}{2^{10}\cdot 5^{10}}} &= 1.
\end{aligned}\right\}$$

These results follow from the fact that the "number of ratios" needed to obtain 10 from unity is given as 10,000,000,000.

When the logarithms of these new numbers are known, the logarithms of other numbers could be deduced.

From this first method it is clear that Napier had devised some arithmetical method of extracting the fifth roots of numbers.

His second method depends on the successive extraction of square roots. Geometrical Means are inserted between the two given numbers, and Arithmetical Means between their logarithms.

e.g.

$$\begin{aligned}
&\text{If} \quad \log 1 = 0,\\
&\text{and} \quad \log 10 = 10{,}000{,}000{,}000,\\
&\text{then} \quad \log \sqrt{10} = 5{,}000{,}000{,}000,\\
&\log \sqrt{10\sqrt{10}} = 7{,}500{,}000{,}000,\\
&\text{etc.}
\end{aligned}$$

This is the method which Briggs followed in the calculation of his Tables, and it was adopted by Vlacq in his continuation of Briggs' work. It is familiar to most teachers of mathematics as a simple method of explaining logarithms to beginners. It must, however, be remembered that in Napier's time the index notation was unknown, and that these results are stated by him in the language of proportionals.

In his third method, it is explained that when

$$\log 1 = 0 \text{ and } \log 10 = 10{,}000{,}000{,}000,$$

a close approximation to the logarithm of any given number may be obtained by finding the number of figures in the result produced by raising the given number to the assumed logarithm of 10.

e.g. The number of figures in the result obtained by raising 2 to the power 10^{10} will be found to be 3010299957.

It follows that when log 1 = 0 and log 10 = 10,000,000,000,

$$\log 2 \text{ lies between } 301\ 029\ 995\ 6 \text{ and } 301\ 029\ 995\ 7.$$

It is clear that if log 10 is taken as 1, instead of 10,000,000 000, the logarithms are obtained simply by dividing those on the former scheme by 10^{10}. Napier chose log 10 as 10,000,000,000, so that he might be dealing only with integers in his logarithms. Briggs changed the logarithm of 10 to unity, and it follows that his results are really logarithms to the base 10 as we now know them.

Extract from Briggs' Table of Logarithms (1624).*

Numeri Absoluti.	Logarithmi.	Numeri Absoluti.	Logarithmi.
16501	4,21751,02642,9403	16534	
	2,63184,8511		
16502	4,21753,65827,7914		
	2,63168,9029		
16503	4,21756,28996,6943		
	2,61352,9567		
⋮		⋮	

H. S. Carslaw.

THE TEACHING OF INDICES AND LOGARITHMS.†

By W. J. Dobbs.

I propose to give you in outline the short course on Indices and Logarithms taken at the present time by the Upper Fifth Form at the Holloway County School. Soon I hope it may be attempted by the Lower Fifth Forms, and, later on, perhaps by kind permission of the examining authority, it may be found possible to include it in the work of the Fourth Forms.

§1. *Index Notation.*—Let a denote any positive numerical quantity, and n any positive integer. Then "$\times a^n$" means "multiplied n times in succession by a." "a^n" alone means "$1 \times a^n$," *i.e.* "unity multiplied n times in succession by a." The index n tells us the number of multiplications.

§2. *Graph of $y=a^x$.*—At present we can assign to x only such values as 1, 2, 3, 4, etc., and the graph consists of a number of isolated points which, however, appear to lie on a curve. Special numerical values such as 1·1, 1·2, 2, are assigned to a, and the corresponding points are plotted on squared paper. If the unit for the x-scale is taken as 1 inch, we see that the ordinate grows in such a way that a movement of one whole inch to the right causes the ordinate to be multiplied by a. Professor Nunn calls the number a the *growth-factor*. Values of x forming the Arithmetical Progression 1, 2, 3, 4, 5,... correspond to values of y in Geometrical Progression.

* The first comma in Briggs' Table of Logarithms represents a decimal point.
In the extracts from the other Tables given above the letters have been spaced for the sake of clearness. This device does not occur in Napier's work.

† A paper read before the London Branch on the 6th of March, 1915.

Now it is clear that we may extend both progressions forwards as much as we please. May we not extend them backwards too? What value of x should be placed before the series 1, 2, 3? and what value of y before the series a, a^2, a^3? As "$\times a^3$" indicates 3 applications of the growth-factor a, what meaning should be assigned to "$\times a^0$"? As "a^3" results from applying the growth-factor a three times to unity, what meaning should be assigned to a^0? Such questions never fail to bring the expected answer.

Referring to the graph, as each step of 1 inch to the right involves a multiplication of the ordinate by a, we are led to interpret each step of 1 inch to the left as involving a division of the ordinate by a. In this way the pupil is led to suggest that "a^{-n}" may be used to denote "unity divided n times in succession by a" and "a^0" as "unity left unchanged."

The graph now consists of a number of isolated points corresponding to integral values of x both positive and negative. All the points appear to be situated on a curve, but, if the curve were drawn, we should at present have no meanings for the intermediate points. We want, for instance, a suitable meaning for "$a^{\frac{1}{2}}$." Now a movement of one whole inch to the right involves a multiplication of the ordinate by the growth-factor a. But a movement of one inch to the right is the same as a movement of two successive half-inches to the right. What growth-factor corresponds to a movement of $\frac{1}{2}$ inch? What growth-factor applied twice in succession produces the same result as a single application of the growth-factor a? The pupil gives the answer—clearly such a growth-factor is the ordinary positive arithmetical square root of a. Two successive applications of the growth-factor $\sqrt{a}$ are together equivalent to one application of the growth-factor a. Thus it appears that $\sqrt{a}$ may be conveniently denoted by $a^{\frac{1}{2}}$.

Now let us consider the graph of $y=(\sqrt{a})^{2x}$. So long as we write for x integral values only, positive or negative, we obtain precisely the same points as before; but we may now write for x

$$\tfrac{1}{2},\ 1\tfrac{1}{2},\ 2\tfrac{1}{2},\ldots,\ -\tfrac{1}{2},\ -1\tfrac{1}{2},\ -2\tfrac{1}{2},\ldots,$$

and we obtain other points situated on the curve which did not appear before.

So we decide to use "$a^{\frac{3}{2}}$" to denote $(\sqrt{a})^3$, *i.e.* "unity multiplied 3 times in succession by $\sqrt{a}$," or "unity multiplied in succession by a and by $\sqrt{a}$." Similarly "$a^{-\frac{5}{2}}$" denotes "unity divided 5 times in succession by $\sqrt{a}$" or "unity divided in succession twice by a and once by $\sqrt{a}$."

Again, consider the graph of $y=(\sqrt[10]{a})^{10x}$. The growth-factor corresponding to a movement of $\frac{1}{10}$ inch to the right is now $\sqrt[10]{a}$. Ten applications of this growth-factor produce the same result as a single application of the growth-factor a, and five applications of the growth-factor $\sqrt[10]{a}$ have the same effect as a single application of the growth-factor $\sqrt{a}$, (*i.e.* $a^{\frac{5}{10}}=a^{\frac{1}{2}}$.) Thus the new graph contains all the points obtained before and many others besides.

Finally, consider the graph of $y=(\sqrt[n]{a})^{nx}$, where n is any positive integer however large. The growth-factor corresponding to a movement of $\dfrac{1}{n}$ inch to the right is now $\sqrt[n]{a}$, and n applications of this growth-factor have the same effect as a single application of the growth-factor a. By sufficiently increasing n, we make the points of the graph as close together as we please, and the graph in this way tends to become a continuous curve. We decide, therefore, to use "$a^{\frac{p}{n}}$" to denote "$(\sqrt[n]{a})^p$" and "$a^{-\frac{q}{n}}$" to denote "$(\sqrt[n]{a})^{-q}$," p and q being positive integers.

Notice that $$a^{-\frac{8}{3}}=a^{-2}\times a^{-\frac{2}{3}},$$

i.e. eight successive divisions by $\sqrt[3]{a}$ are together equivalent to two successive divisions by a followed by two successive divisions by $\sqrt[3]{a}$.

"$10^{1\cdot357}$" denotes "unity multiplied in succession once by 10, three times by $\sqrt[10]{10}$, five times by $\sqrt[100]{10}$ and seven times by $\sqrt[1000]{10}$."

§3. *The Fundamental Law of Indices.*—The statement

$$a^m \times a^n = a^{m+n},$$

when m and n are positive integers, merely asserts that m applications of the growth-factor a followed by n applications of the growth-factor a amount to $(m+n)$ applications of the growth-factor a. The same law is true for negative as well as positive integers. For instance,

$$a^3 \times a^{-7} = a^{-4}.$$

Here unity is multiplied 3 times in succession by a, and these 3 multiplications by a are followed by 7 divisions by a, resulting in 4 divisions by a. The -4 is obtained by adding the 3 and the -7.

After a few numerical exercises of this description, it is clear that any other integers, positive or negative, may be treated in the same way, and the law is true for all integral values of m and n, whether positive or negative.

Let us now consider fractional indices,—for instance

$$a^{\frac{2}{3}} \times a^{\frac{3}{4}}.$$

We know that $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$. Also 2 multiplications by $\sqrt[3]{a}$ are together equivalent to 8 multiplications by $\sqrt[12]{a}$, while 3 multiplications by $\sqrt[4]{a}$ are together equivalent to 9 multiplications by $\sqrt[12]{a}$. Thus

$$a^{\frac{2}{3}} \times a^{\frac{3}{4}} = a^{\frac{17}{12}},$$

and the process of obtaining the $\frac{17}{12}$ is precisely that of adding together the two fractions $\frac{2}{3}$ and $\frac{3}{4}$.

Other numerical examples of positive and negative fractions are considered, and such are more illuminating to the ordinary boy than a proof in general terms, while the process involved is clearly shown to be precisely that of adding together m and n. Thus in all cases

$$a^m \times a^n = a^{m+n}.$$

§4. *Another Form of the Fundamental Law of Indices.*—The above law may also be expressed thus:

$$a^m \div a^n = a^{m-n};$$

for, division by a^n is in all cases the same as multiplication by a^{-n}.

§5. *Another Law of Indices.*—We proceed to consider expressions of the type $(a^m)^n$. Integral values of m and n present no difficulty. For instance,

$$(a^{-3})^2 = a^{-6} = (a^3)^{-2},$$

i.e. 2 batches of 3 divisions by a together amount to 6 divisions by a.

Again, $$(a^{-3})^{-2} = (a^3)^2 = a^6,$$

i.e. 2 divisions by a^{-3} are together equivalent to 2 multiplications by a^3, and also to 6 multiplications by a.

Now let m have any value, integral or fractional, positive or negative, and let q be any positive integer. Since q multiplications by $a^{\frac{m}{q}}$ are together equivalent to a single multiplication by

$$a^{\frac{m}{q}+\frac{m}{q}+\dots\text{ to } q \text{ terms}},$$

i.e. by a^m, we see that $a^{\frac{m}{q}} = \sqrt[q]{a^m}$ in all cases, so long as q is a positive integer.

Now
$$(a^m)^{\frac{3}{q}} = 1 \times \sqrt[q]{a^m} \times \sqrt[q]{a^m} \times \sqrt[q]{a^m}$$
$$= 1 \times a^{\frac{m}{q}} \times a^{\frac{m}{q}} \times a^{\frac{m}{q}}$$
$$= a^{\frac{3m}{q}},$$

i.e. 3 multiplications by $\sqrt[q]{a^m}$ are together equivalent to a single multiplication by $a^{\frac{3m}{q}}$.

Again,
$$(a^m)^{-\frac{4}{q}} = \frac{1}{\sqrt[q]{a^m} \times \sqrt[q]{a^m} \times \sqrt[q]{a^m} \times \sqrt[q]{a^m}}$$
$$= \frac{1}{a^{\frac{m}{q}} \times a^{\frac{m}{q}} \times a^{\frac{m}{q}} \times a^{\frac{m}{q}}}$$
$$= a^{-\frac{4m}{q}},$$

i.e. 4 divisions by $\sqrt[q]{a^m}$ are together equivalent to a single division by $a^{\frac{4m}{q}}$ or to a single multiplication by $a^{-\frac{4m}{q}}$.

In this way we prove that if p is any integer, positive or negative, and q any positive integer, then, whatever be the value of m,

$$(a^m)^{\frac{p}{q}} = a^{\frac{pm}{q}}.$$

Thus the law $(a^m)^n = a^{mn}$ is true in all cases.

Referring again to the graph in which a movement of 1 inch to the right corresponds to the application of a growth-factor a, a movement of m inches corresponds to the growth-factor a^m, and a movement of mn inches corresponds to the growth-factor a^{mn}. Thus the law of indices here considered expresses the fact that the nth power of the growth-factor which corresponds to a movement of m inches is the growth-factor which corresponds to a movement of mn inches.

§ 6. *Simple Properties of the Function* a^x.—Let a denote any positive number, not necessarily an integer, and let $y = a^x$. The addition of 1 to the value of x applies the growth-factor a to the value of y, and, if n is any positive integer, the addition of $\frac{1}{n}$ to the value of x applies the growth-factor $\sqrt[n]{a}$.

First, let $a > 1$. Then $\sqrt[n]{a}$ is also > 1 however great n may be. As x increases from 0, y continually increases from 1 and is positive; as x changes from any negative value towards 0, y continually increases towards 1 and is positive. Whenever the ordinate moves in the positive direction of the x-axis, it continually increases, remaining positive. If the ordinate moves in the negative direction of the x-axis, it continually decreases, remaining positive.

Suppose now that a greater and greater *positive* value is assigned to x, the value of y continues to increase. There is no limit to the magnitude of the value which may be assigned to x. Is there any limit to the corresponding positive magnitude of y?

As $a > 1$, we may write $a = 1 + b$, where b is a positive numerical quantity. We know that $(1+b)^2 > 1 + 2b$. Multiplying by $(1+b)$, $(1+b)^3 > 1 + 3b$, and so on. Hence, if m is any positive integer, $(1+b)^m > 1 + mb$,

$$\text{i.e. } a^m > 1 + mb.$$

Hence a^x can be made as great as we please by sufficiently increasing x. In other words, there is no limit to the possible positive magnitude of y.

Now consider *negative* values of x, and let $x=-X$. Then X is positive and $a^x=\frac{1}{a^X}$. Now a^X can be made as great as we please by sufficiently increasing X. Hence $\frac{1}{a^X}$ can be made as small as we please, *i.e.* a^x can be made as small as we please, by assigning to x a sufficiently great negative value. The general character of the graph of $y=a^x$ is now clear.

Secondly, let $a<1$. Then we may write $a=\frac{1}{\alpha}$, where $\alpha>1$. On reversing the direction of the x-axis the graph of $y=a^x$ becomes identical with that of $y=\alpha^x$.

In this connection we consider the summation to infinity of a Geometrical Progression.

§ 7. *The case in which* $a=10$.—As the common system of numeration is based on the number 10, the case in which $a=10$ becomes of special importance. In tabulating the values which the function 10^x assumes for different values of x, it is sufficient to allow x to range in value between 0 and 1. A few numerical examples are given.

An approximation to the value of 10^x for any value of x between 0 and 1 may be obtained in this way:—By ordinary arithmetic we calculate, say to three significant figures,

$$\sqrt{10},\quad \sqrt{10\sqrt{10}},\quad \sqrt{\sqrt{10}}.$$

We then tabulate approximate values of 10^x for the following values of x:

$$0,\quad 0{\cdot}25,\quad 0{\cdot}50,\quad 0{\cdot}75,\quad 1,$$

and draw the corresponding graph. By means of this graph we can express any given value of y intermediate in value between 1 and 10 in the form 10^x, reading off the approximate value of x. For instance, 3 appears to be approximately $10^{0{\cdot}48}$, indicating that multiplication by 3 is approximately equivalent to 4 multiplications in succession by $\sqrt[10]{10}$ followed by 8 multiplications by $\sqrt[100]{10}$.

§ 8. *Common Logarithms.*—When $N=10^n$, n is called the *Common Logarithm*, or the *Logarithm to base* 10, of N. To illustrate this change in the point of view, it is convenient to redraw the graph, interchanging the axes. This graph constitutes an approximate table of logarithms of all numbers ranging from 1 to 10. For instance, we read

$$\log 7{\cdot}4=0{\cdot}87 \text{ approximately.}$$

This means that multiplication by 7·4 is approximately equivalent to 8 successive multiplications by $\sqrt[10]{10}$ followed by 7 successive multiplications by $\sqrt[100]{10}$.

We deduce that

$$\log 740=2{\cdot}87 \text{ approximately,}$$
$$\log 0{\cdot}0074=\bar{3}{\cdot}87 \text{ approximately,}$$
$$\text{etc.}$$

Again, we read

$$\left.\begin{array}{l}\log 2=0{\cdot}30\\ \log 3=0{\cdot}48\end{array}\right\}\text{approximately,}$$

and interpret these statements as before. Then we add the two logarithms, and the pupil anticipates that the result should be approximately log 6, for multiplications by 2 and 3 are both included. The graph checks the inference.

We work a few exercises, using the graph as a table of logarithms, and realise that, to be practically useful, the logarithms of numbers must be

calculated to a higher degree of accuracy. Then the use of four-figure tables of common logarithms is explained, and a little drilling in reading them suffices. For instance,

$$\log 8{\cdot}274 = 0{\cdot}9177.$$

The meaning is that multiplication by 8·274 is approximately equivalent to successive multiplications 9 times by $\sqrt[10]{10}$, once by $\sqrt[100]{10}$, 7 times by $\sqrt[1000]{10}$, and 7 times by $\sqrt[10000]{10}$. The *number* commences with a digit followed by the decimal point, and the *logarithm* commences with 0 followed by the decimal point.

Also, $$\log 827{\cdot}4 = 2{\cdot}9177,$$

for two multiplications by 10 are also required.

Also, $$\log 0{\cdot}8274 = \bar{1}{\cdot}9177,$$

for 1 division by 10 is required. No mention is made of the terms *characteristic* and *mantissa*, and no rules are given for determining the characteristic. Not until this stage do we consider formal proofs of the three laws of common logarithms.

In working examples on the use of four-figure tables, the logarithms are written down in column, and at first each line is labelled. The process of subtraction is that of "complementary addition."

Thus

$$\begin{array}{l} 1{\cdot}3227 \\ \underline{\bar{2}{\cdot}3954} \\ 2{\cdot}9273 \text{ (subtracting)} \end{array}$$

and 3, 5 and 7 carry 1, 10 and 2 carry 1, 4 and 9 carry 1, $\bar{1}$ and 2.

I am a little uncertain whether to introduce generally the following simple device by which a subtraction is turned into an addition :

Suppose that we wish to subtract a logarithm whose integral part is I and fractional part F.

Then $$-\{I+F\} = +\{-(I+1)+F'\},$$

where $$F+F'=1.$$

The fraction F' may be called the complement of the fraction F. The digits of F' are derived from those of F by completing each digit of F up to 9 except the digit on the extreme right, which must be completed up to 10. Hence the rule—*Add* 1 *to the integral part of the logarithm and change the sign of the result; replace the fractional part of the logarithm by its complement.*

For instance, to subtract 2·6384 add $\bar{3}{\cdot}3616$,
to subtract $\bar{2}{\cdot}5821$ add 1·4179.

§ 9. *Logarithms to Different Bases.*—Let any positive number a, greater than 1, be chosen as the base of logarithms. Then a^n increases when n increases, and decreases when n decreases. Hence $\log N$ increases when N increases, being negative when $N<1$ and positive when $N>1$. Formal proofs of the three laws of logarithms to base a are now given, and a fourth law

$$\log_b N = \frac{\log N}{\log b}$$

is established. Thus it appears that, to change a table of logarithms from the base adopted to a new base b, every logarithm must be multiplied by the same numerical factor $1/\log b$. The ratio of the logarithms of any two given numbers is the same whatever be the base. In dealing with logarithms to different bases, it is convenient to transfer them all to some standard base which need not be named.

Thus, $$\log_b a \times \log_a b = \frac{\log a}{\log b} \times \frac{\log b}{\log a} = 1.$$

§ 10. *The Distributive Law of Indices.*—Let a and b be any positive numbers, not necessarily integers, and let n have any value, positive or negative, integral or fractional.

Then
$$\begin{aligned}\log(ab)^n &= n\log(ab)\\ &= n(\log a + \log b)\\ &= n\log a + n\log b\\ &= \log a^n + \log b^n\\ &= \log(a^n b^n);\\ \therefore\ (ab)^n &= a^n b^n,\end{aligned}$$

i.e. *An index applied to a product is distributive over the factors of the product.*

Also,
$$\begin{aligned}\left(\frac{a}{b}\right)^n &= (a\,.\,b^{-1})^n\\ &= a^n\,.\,(b^{-1})^n\\ &= a^n b^{-n}\\ &= \frac{a^n}{b^n},\end{aligned}$$

i.e. *An index applied to a fraction is distributive over the numerator and denominator of the fraction.*

W. J. Dobbs.

REVIEWS.

Combinatory Analysis. By Major P. A. MacMahon. Vol. I. Pp. xx + 300. 15s. net. 1915. (Cam. Univ. Press.)

Major MacMahon is a past-master in every kind of symmetrical algebra, and this work will be welcomed by all who enjoy "tactic" permutations, the theory of forms, and so on. It is full of originality and elegance in the best sense of the term.

After an introductory chapter on Symmetric Functions, the first section deals with Distributions, and particularly with Hammond's operators d_λ, D_λ. We have also operators δ_λ, Δ_λ operating on functions expressed in terms of homogeneous products, instead of elementary symmetric functions. Two laws of symmetry are deduced by means of them.

Section II. is on Separations; for instance $(ab)(c)(d)$ is a separation of $(abcd)$. Here we have various illustrations of symmetry, one-one correspondence, and reciprocity. Girard's (generally known as Waring's) theorem is generalised, new differential operators are introduced, and there is a chapter on a calculus of binomial coefficients.

Section III. is on permutations. Here, as elsewhere, the reader must be struck by the author's power of constructing and analysing generating functions which, by their formal expansion, give desired enumerations. The power of the lattice method is also illustrated. On p. 97 we have the statement of what Major MacMahon calls "the master theorem." Stated briefly, this is:

"If $(X_1, X_2, \ldots X_n) = (a_{nn})(x_1, x_2, \ldots x_n)$, the symbol (a_{nn}), meaning a matrix, making X_i a linear function of $(x_1, \ldots x_n)$, then the coefficient of $x_1^{\xi_1} x_2^{\xi_2} \ldots x_n^{\xi_n}$ in the expansion of $X_1^{\xi_1} X_2^{\xi_2} \ldots X_n^{\xi_n}$ is equal to that of the same term in the expansion of $1 \div [(1 - a_{11}x_1)(1 - a_{22}x_2) \ldots (1 - a_{nn}x_n)]$, which means the expansion of $1 \div (1 - a_{11}x_1)(1 - a_{22}x_2) \ldots (1 - a_{nn}x_n)$ and the subsequent change of such a coefficient as $a_{11}a_{22}$ into the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ of the matrix (a_{nn})."

Applications of this are made to displacements, self-conjugate permutations, lattice permutations, and so on.

Section IV. is on the compositions of numbers. Here there is much original work on multipartite numbers with applications to lines of route, Cayley's "trees," and so on. Graphical methods now come into prominence. On p. 203 we are introduced to a new set of symmetric functions, or rather two conjugate sets of them denoted by a new notation; namely, a symbol with a suffix which is a partition. This seems an important idea; special examples are:

$$a_{11}=a_1^2-a_2=h_2,$$
$$a_{111}=a_1^3-2a_1a_2+a_3=h_3,$$

etc., a well-known set of functions that occur in the theory of invariants.

Section V. is on distributions on a chess-board; under this come discussions of the "problème des ménages," and that of Latin squares. Section VI. is on the enumeration of partitions of multipartite numbers. It is not very difficult, after inductive experiment, to find a generating function; the interesting thing is to see how deductions are made from it without any appearance of effort.

Finally we have tables of (i) homogeneous product sums in terms of monomial symmetric functions; (ii) products of elementary s.f. in terms of monomial s.f.; (iii) monomial s.f. in terms of elementary s.f.—all up to weight 6; (iv) the simplest cases of distribution functions; (v) binomial coefficients; (vi) composition enumerations; (vii) the a and h s.f. above alluded to; (viii) symmetric tables associated with partitions, up to weight 6.

Throughout the book the author wisely begins each new topic with a particular example. In this way we are able to gain some idea of his induction, and are greatly helped in appreciating his general theorems. It may be added that, considering the variety of problems discussed, the general consistency of the work is remarkable. All who are interested in this branch of Mathematics will be eager to see the rest of the treatise.

One result of reading this book has been to confirm my admiration for the genius of Hammond, in his particular line. He was the first to show how to make full use of linear differential operators, and his reduction of redundant generating functions was wonderfully ingenious. No doubt he owed much to Sylvester; but his name will always be remembered in connection with the algebra of forms. G B. MATHEWS.

Differentialgleichungen und Variationsrechnung. (Dritter Band der Lehrbuch der Differential- und Integralrechnung.) By J. A. SERRET, nach Axel Harnack's Übersetzung. Vierte und fünfte Auflage, bearbeitet von Georg Scheffers. Pp. xiv + 735. 1914. (Leipzig: B. G. Teubner.)

This treatise has received a good deal of emendation in the course of successive revisions, and little remains of Serret's work except the examples and general outline. It is not a book for the beginner, especially if his interest is in the direction of Applied Mathematics, and in devices for the actual solution of differential equations. It is true that he will find here the usual methods of solution, but they are introduced incidentally and with few illustrations; while no examples are given as exercises for the reader, as is usual in English text-books. Even when methods of solution are given, they are not always the best available from the practical standpoint, as in the case of Pfaff's equation, and still more noticeably in the case of linear simultaneous equations (§§ 871, 773). But the advanced student or the teacher to whom the ordinary methods of solution are quite familiar, and who is interested rather in a rigorous and detailed account of the general theory of differential equations, will find the book useful and attractive.

On the whole, the text is clear and as easy to follow as the nature of the subject will allow; except, perhaps, in the treatment of Lie's group-theory, which would hardly be intelligible to a reader with no previous knowledge of the subject. The author shows good judgment in working out in detail the case of two equations in "normal form" (Chap. 2) before taking the more general case of n equations, thereby materially lessening the difficulty of following the discussion. With good judgment, again, he marks certain portions which may be omitted by a student who wishes to confine himself to the theory of real variables in the first instance.

The author unfortunately neglects some good opportunities of giving an example for which a theorem under discussion is inapplicable. Such an example seldom fails to give a sense of pleasure to a reader, and to bring home to him the real need for the restrictions stated in the theorem more forcibly than any example which satisfies the conditions of the enunciation. The theory of partial differential equations of higher order than the first finds no place in the book. A discussion of such equations might well have replaced the chapter on Calculus of Variations, which makes no attempt at rigour, and is far below the high standard set in the rest of the work.

There is a full index, and the text seems very free from misprints. An appendix, containing references and historical notes, makes an interesting and valuable addition to the book. H. H.

Physics of the Household. By C. J. Lynde. 5s. 6d. net. 1915. (Macmillan, N.Y.)

Analysis—pp. 1-78, Mechanics; pp. 79-168, Heat; pp. 169-245, Electricity and Magnetism; pp. 246-273, Light; pp. 274-288, Sound; pp. 289-302, Mechanics; pp. 303-313, Tables; Index.

It is probable that many teachers of Mechanics, as well as of general physics, will find this book a useful supplement to their regular texts. A few years ago text-books on Mechanics were hopelessly unreal. More recently, reality has sometimes been secured at the expense of intelligibility. Few boys can have first-hand acquaintance with the intricacies of modern machinery, and vague references do not tend to clear-cut thought.

Professor Lynde seeks to avoid both errors by the choice of comparatively familiar pieces of mechanism, shown in adequate diagrams. Thus we find various principles of mechanics illustrated by reference to such things as coffee-mills, different systems of water-supply, vacuum cleaners, etc. From this point of view, the book is very successful, and should, in particular, be of assistance in preparing for "Second Division" and similar examinations in general science.

From the theoretical standpoint one or two minor criticisms may be made. It seems, to the reviewer, unwise to use such expressions as "a pound of force," "the distance the force moves," "a body when placed in a liquid loses weight," and "density . . . is the weight of unit volume." These, however, are easily corrected and do not seriously impair the usefulness of the book. For class purposes, the addition of a set of answers to the examples is desirable. A. F. H.

Exercises in Algebra (including Trigonometry). By T. Percy Nunn, M.A., D.Sc. Part I., with Answers, 421 pp. 4s. Part II., with Answers, 551 pp. 6s. 6d. (Longmans.)

To form an adequate opinion on the merits of these two volumes it is necessary that they should be read in connection with the companion volume on *The Teaching of Algebra* by the same author, wherein is to be found "a discussion of the principles upon which the author has selected and presented the subject-matter of the exercises" and "suggestions as to the order in which the exercises should be taken, and a full statement of the preparatory teaching presupposed in each."

On reading the companion volume first, one is left with the idea that a great method has been considered, but one more or less impracticable for everyday schoolwork: this impression is banished entirely on considering the exercises, where the method is worked out in minute detail, and proved to be perfectly feasible, at least to those who are in thorough agreement with the author's pedagogic standpoint.

Volume I. consists of three sections: on non-directed numbers, directed numbers, and logarithms respectively. Section I. starts with the consideration of Algebra as "Shorthand," and the further development proceeds from the starting point of the formula. The method of development is graduated and exceedingly good; but the idea of Algebra as a shorthand simply is very dangerous. Thus on page 1, we find C standing for "the total cost." That this is not a clerical slip on the part of the author is evident from the companion volume on the method of teaching. My experience

shows that unless a student is taught from the first that letters are used simply instead of numbers, just as much so as the figures 1, 2, 3, nothing but harm results to the lower half of the class, and even to the better boys the theory of units is a dead letter at a later stage. Also, the use of the formula as a starting-point, instead of the simplest type of equation—for the formula is only a relatively hard equation for the beginner—is to be deprecated, and seems to be advocated on the score of utility, interest, and intuition.

When a boy or girl is struggling with the beginnings of Algebra, surely such formulae as $W=Kbd^2l$, and questions founded thereon, are out of place. They cannot be defended on the score of intuition, for no child can have any intuitive knowledge of the "breaking weight in cwt." of "a wrought-iron bar 10 feet long and 2 inches square." I feel strongly that even in Practical Mathematics for Technical Students in mixed classes all such questions are bad, in that they are not intuitional to *all* the students: and it is to be remembered that this volume is designed for secondary school use, or, at least, for students under sixteen years of age. Apart from this point of pedagogic principle, there is nothing too extravagant in the way of praise to be said of the section.

Section II. on directed numbers is just as good, the author laying his finger (with unerring accuracy) on the point of failure or difficulty attending the introduction of negatives: namely, the preliminary teaching, which gave the student the idea of the numerical relation of greater or less between two numbers instead of the correct idea of order. Negative indices are happily introduced by means of "standard form": logarithms by the use of a logarithmic curve formed from a growth factor and a Gunter scale, in Section III. I think this latter is incorrect: boys should be taught the use of tables, as tools only, very early in arithmetic, the explanation can then very well wait until they have studied some treatment of incommensurables, such as is given in Prof. Bowley's *Pure Mathematics*. This, again, is however a matter of pedagogics only.

A boy or girl of sixteen, who has worked through Volume I., will, in addition to a general grasp of principles, have acquired a very large stock of general knowledge: whether the latter will help or hinder the former must be put to the proof of practice.

Volume II., intended as a further course from sixteen onwards, is free from any objection on the score of presentment. Provided with a sound knowledge of fundamentals, the student is practically independent of pedagogics. In this volume there is a sufficiency of Spherical Trigonometry (perhaps over-elaborated by the introduction of too much technical matter, such as Sanson's Net, etc.): complex numbers, the circular and hyperbolic functions and limits get full and careful treatment: from the latter is developed the notion of a gradient, and hence follow the fundamental principles of the calculus. Again a matter of pedagogics—Should not the notion of a gradient come much earlier? Every boy (and some would say girl) of sixteen should know how to differentiate and integrate (not necessarily so-called) a power of x. A section on Statistics closes the volume: this section is necessarily difficult.

Summing up, these books without doubt develop a grand system of mathematical training, but I hardly think one which will be acceptable to the ordinary teacher, who teaches for a living and not with the enthusiasm of a Dr. Nunn: without this enthusiasm the work of the teacher would become irksome. Unfortunately for authors, this is a fence at which the good horse, Reform, has often fallen.

Norman's Arithmetic for Schools. By J. S. and F. K. NORMAN. Pp. xvi + 278. 2s. With Answers, 2s. 6d. 1915. (The Year Book Press.)

The Preface of this work contains the following statement: "The Book work is intended primarily for Classical men who have had but little teaching in Arithmetic, and are unable to explain to the pupils the reasons for the various processes they adopt." The parents of Mr. Norman's pupils will surely be interested to learn that the teaching of fundamental Arithmetic is delegated to men who, as he candidly admits, are scarcely competent to

undertake it. Such men may find this volume stimulating and instructive, but we fear it will require more than a constant perusal of it to render them efficient.

The subject-matter is simply a concise summary of the verbal explanations and questions that any qualified teacher would give to his class. As such it is interesting and on the whole clearly expressed, but we have failed to notice anything particularly novel or striking.

It is sometimes a little difficult to catch the drift of the questions; *e.g.* those on page 103 about keys. The first would appear to be originally propounded by the author of "How many beans make five?"

The examples seem very well chosen, *e.g.* those on reduction and decimals; there are none of the huge masses of figures to manipulate which distract the pupil's attention from his argument. The papers at the end of the book are singularly free from the monstrosities which so frequently figure in such collections, and afford an excellent choice of problems.

The method given for compound multiplication on page 27, in place of the notoriously unsatisfactory process "multiply by 10 and by 10 again, etc.," will be novel and welcome to many. The section on decimals leaves little to be desired, but might with advantage come rather earlier. Nearly the whole of the usual syllabus is covered. Though questions on Stocks and Shares occur in the Problems, no reference appears to be made to them in the text. The subject of Practice is entirely omitted—a very serious defect in an otherwise useful volume. Surely also Averages and Compound Interest deserve some attention in a work the latter Papers of which can be intended only for fairly advanced pupils, *e.g.* Paper LXXVIII.

We must emphatically condemn such astonishing statements as: 12 in. $\times$ 12 in. $=$ 144 sq. in. (page 35), a rugby football ground is 110 $\times$ 75 yards (page 37), and the like, which abound in the section on Areas and Volumes.

A Shilling Arithmetic. By W. M. Baker and A. A. Bourne. Pp. xiv + 192. 1s. With Answers, 1s. 4d. 1915. (Bell & Sons.)

The most noteworthy feature of this little volume is its extraordinarily good value for the price. It embodies a complete course, and nothing essential is omitted. It even includes a variety of examples on the mensuration of the Cone, Sphere, Cylinder, etc.

The authors show that they are fully acquainted with the usual pitfalls to which the student falls a victim, and the book abounds with sound practical advice (oft repeated) on matters of detail. Algebraic symbols are freely used whenever a solution can be thus rendered more clear and convincing, *e.g.* in questions on Proportional Parts. The chapter on negative quantities hardly serves any useful purpose in a work of this kind and might well be omitted. Neither the definitions nor the illustrative remarks about a farmer and his cows are altogether satisfactory.

Emphasis is laid on the fact that all questions on interest and percentages can be solved by the Unitary Method. A large number of these, however, can be calculated far more rapidly and accurately by "Practice" methods, similar to that given for Compound Interest on page 141.

The book contains a sufficiency of good examples, oral and other, and a large number of revision papers.

Workshop Arithmetic. By Frank Castle. Pp. viii + 172. 1s. 6d. 1915. (Macmillan.)

This little book is written for Engineers and those engaged in the Building Trades. It begins with a section of 60 pages consisting of a short though adequate revision of all that is necessary in order to tackle successfully the very practical and complete course of Mensuration which follows.

J. M. Child.

A method for calculating that part of the recoil momentum of a gun which is due to the action of the gas after the projectile leaves the muzzle. By Wm. S. Franklin, Professor of Physics, Lehigh University. Reprinted from the Journal of the Franklin Institute, May 1915. Pp. 559-577.

The problem is of some importance in the practice of Artillery. The treatment is impracticable by the methods of pure Pneumatical Dynamics. Stated in

its elementary simplicity, the question is to discuss the motion of compressed air in a closed tube, when one end is suddenly uncorked, so there is a certain analogy with the action of a toy spring gun, when the inertia of the spring is taken into account; also with the impact of an elastic cylindrical rod, to which the graphical solutions have been applied which are given in the memoir.

Chart Atlas of Complex Circular and Hyperbolic Functions. By A. E. KENNELLY, Sc.D., A.M., Professor of Electrical Engineering in Harvard University. 21×20 inches. 12s. net. 1914. (Harvard University Press and Oxford University Press.)

"I have no use for the Hyperbolic Function," the mathematician was saying not long ago; but these elaborate tables and diagrams would not have been drawn up unless there was a demand for them among electricians.

The object is to employ the circular and hyperbolic function for the expression and graphical representation of the real and imaginary part of $\sin(x+yi)$ and $\cos(x+yi)$.

Weir's Azimuth Diagram seems unknown to the author; this is a similar chart, composed of a system of orthogonal confocal ellipses and hyperbolas, and satisfies the same purpose, although designed originally to solve graphically a problem of Navigation.

With steam and the chronometer, the chief requirement at high speed is the determination of the variation of the compass, so as to steer a correct course, and this requires a frequent observation of the Sun's altitude and its conversion into azimuth.

Weir's Diagram (Potter, in the Minories, 2s. 6d.) should be a wall diagram of the mathematical class-room, and some of the Kennelly diagrams may be recommended for the same purpose, apart from their primary utility in electrical calculation.

The construction of such a diagram should form a useful exercise on the drawing board for a mathematical student, provided he is not allowed to become fascinated with this sort of work, to the neglect of the Calculus and other analytical thought. G. GREENHILL.

Plane Trigonometry. By H. S. CARSLAW. Pp. xviii + 293 + xi. 4s. 6d. 2nd Edition. 1915. (Macmillan & Co.)

Solutions of the Questions in Plane Trigonometry. By H. S. CARSLAW. Pp. 179. 6s. 6d. net. 1915. (Macmillan & Co.)

The merits of Prof. Carslaw's little treatise, with its more or less conservative treatment of the subject for the benefit of the student of sufficient ability to proceed forthwith to the assault of a standard volume such as that of Prof. Hobson, have been sufficiently recognised to justify the issue of a second edition. It is now accompanied by a small volume of Solutions to the Examples in the text-book, which are carefully and lucidly done, and should be of value to all private students, and to many teachers whose equipment has not brought them abreast of work which they are unduly called upon to undertake.

Subjects for Mathematical Essays. By C. DAVISON. Pp. x + 160. 3s. 6d. 1915. (Macmillan & Co.)

Dr. Davison's collection of papers will be welcome to those teachers who are not very familiar with the aim and scope of the modern "Bookwork Paper," which is now an essential feature in the scholarship papers set at certain colleges. The papers are carefully constructed, and in the hands of a judicious teacher should be of great value for the purpose for which they are designed.

Mathematical Papers for admission into the Royal Military Academy and the Royal Military College, for the years 1905-1914. Edited by R. M. MILNE. 6s. 1915. (Macmillan & Co.)

Ditto, for September-November, 1914. Pp. 27. 1s. net. 1915. (Macmillan & Co.)

This familiar collection has been arranged by Mr. Milne of the Naval College, Dartmouth, and—what will to many enhance its value considerably—

is issued with answers. The booklet will be useful to those who desire to have the latest of these sets of questions, and it is also published with answers.

Improved Four-Figure Logarithm-Tables. Multiplication and Division made easy. By G. C. M'LAREN. Pp. 27. 1s. 6d. net. 1915. (Cambridge University Press.)

"When multiplying together any two numbers of four figures or dividing one by the other with the aid of this book, the smallest possible error can never be more than 1 in the fourth figure of the true answer when cut down to four figures. By the four-figure logarithm card, on the other hand, an error of as much as 5 in the fourth figure is liable to appear." Hence the *raison d'être* of Mr. M'Laren's book of tables. The last figure is shown to the nearest third, *e.g.* 1810, 1838; 1866. Thumb-indexing facilitates rapid work. Experience alone will show whether calculators will prefer the new lamp to the old.

Worked Exercises in Elementary Geometry. By F. C. GILLESPIE. Pp. iv+136. 1914. (Oxford Univ. Press.)

This is a collection of riders in the main from the last nine years' papers set for Responsions and Pass Moderations, but it naturally covers in addition the syllabus for many other examinations. Its object is to "aid the student in interpreting any geometrical question in terms of the established book-work, in order that he may see at once how any particular exercise is to be solved." It should be of the greatest value to the private student who has the self-restraint to use solutions in the proper way.

Annuaire pour l'an 1915, publié par le Bureau des Longitudes, avec Une Notice scientifique. Pp. vii+764+173+58. 1 fr. 50 c. net. 1915. (Gauthier-Villars.)

It is interesting to note that the appearance of this well-known annual has not been appreciably delayed by the war of 1914-1915. It contains detailed tables relative to Astronomy, Geography, Statistics, Metrology, Meteorology, and Money. Next year will be the turn of Physical and Chemical Tables. The "Notice" is by M. G. Bigourdain, and is entitled, "Les Méthodes d'Examen des Miroirs et des Objectifs." As might be expected from the hands to whose preparation it was entrusted, this is a masterly compendium of the subject.

A Treatise on Statics. By G. M. MINCHIN. Vol. II. 5th Edition. Revised by H. T. GERRANS. Pp. vii+369. 10s. 6d. 1915. (Clarendon Press.)

The changes and omissions in the present edition were made before Prof. Minchin's death with the consent of the author and others interested. The omitted portions will, it is hoped, "form the basis of a separate work." An account of the author's "Recent Researches in Spherical Harmonics" is included. The Appendix contains a fine collection of over 500 examples from various sources, mainly from Oxford Examinations, and hints for the solution of some are added.

Opera Matematiche di Luigi Cremona. Tomo Secondo. Pp. 459. L. 25. 1915. (Hoepli, Milan.)

Among the more interesting of the papers in this second volume of the collected writings of Cremona are: A review of Poudra's two-volume edition of the works of Desargues; a bibliographical article on the Theory of Conics, based on certain papers published by Chasles in the *Comptes rendus*, 1864; a sketch of the History of Perspective, ancient and modern, in which a glowing tribute is paid to the *aureo opusculo* of the author of the *Methodus incrementorum*, and to the remarkable simplicity of the proofs in Brook Taylor's *Linear Perspective*, in which are solved "all the more important direct and inverse problems of perspective"; an exposition of Brook Taylor's *Principles*, published under the anagram Marco Uglieni; the prolegomena to a Geometrical Theory of Surfaces (1866), extending over more than 100 pages; and sundry papers on gauche curves and surfaces.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette.*

Dear Sir,—I was pleased to read the letter from Dr. Sommerville in the last number of the *Mathematical Gazette.* The new point which Dr. Sommerville raises in reference to the proof of Desargues' Theorem is a matter of general interest to those who have studied or written on Projective Geometry, and involves the question how far it is justifiable to combine different methods of treating the subject. The more important theorems may be deduced by at least five different methods: (1) by conical projection, wherein the theorems for the conic are deduced from those for the circle; (2) by the analogous method of homology or plane perspective; (3) by the use of ratios and anharmonic properties involving the idea of the ratios of lengths—in this method Ceva's, Menelaus' and Carnot's theorems are of primary importance; (4) by using the anharmonic properties of ranges and pencils to prove directly the properties of conics; (5) by commencing with certain fundamental axioms and developing the subject independently of the results obtained by Euclidian geometry. The point at issue is how far it is justifiable to combine these different methods. The natural instinct of a writer acquainted with the subject is to write distinct sections dealing with the subject from these different points of view. But the question arises whether this is the best way of presenting the subject to the learner—and it is for the learner that the majority of mathematical books must be written. After careful consideration I came to the conclusion that it was desirable to follow in the steps of Professor Cremona and more or less combine the different methods of treating the subject. For the learner, as a general rule, the easiest method is the best, and the time available for the study of Pure Geometry in the career of a student for mathematical honours is so limited that an independent consideration of the subject of projective geometry from every point of view is impossible. The only practical method for placing the different points of view before the student seems to me to lie in supplying alternative proofs of the more important theorems illustrating the different methods. To confine the attention of the student to one of these would limit too much his view, and would render him unable to approach many problems which, considered from another point of view, present little or no difficulty. It should be borne in mind that theorems involving the ratios of distances are just as much projective as those relating to the concurrency of straight lines, etc. On the ground that new axioms should be introduced only when they cannot be dispensed with, I have abstained from their introduction till treating of one to one geometrical correspondence.—I am, Yours truly,

John L. S. Hatton.

East London College (University of London),
3rd May, 1915.

Dear Sir,—A correspondent sent me some time ago a suggestion concerning the proofs given in my Tract, *The Integration of Functions of a Single Variable,* of the fundamental theorem concerning unicursal curves. The suggestion referred to the case in which some of the singular points are ordinary cusps. I took note of the suggestion and wish to incorporate it in a new edition of the Tract. But I have unfortunately lost the letter, and am unable to remember to whom the suggestion is due. I should be much obliged if my correspondent would communicate with me again, if he should happen to see this note.—Yours truly,

G. H. Hardy.

Trin. Coll., Cambridge.

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